Purification of single-photon entanglement with linear optics

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We show that single-photon entangled states of the form $|0\rangle|1\rangle + |1\rangle|0\rangle$ can be purified with a simple linear-optics based protocol, which is eminently feasible with current technology. Besides its conceptual interest, this result is relevant for attractive quantum repeater protocols.

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Single-particle entanglement of the form $|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B$ may be the simplest form of entanglement [1]. It corresponds to a single particle that is in a superposition state of being in location (mode) A and of being in location B. The particle can be a freely propagating photon as in proposals for linear-optics quantum computing [2], but also e.g. a single excitation in an atomic-ensemble based quantum memory, i.e. a stored photon [3]. Single-photon entanglement has been teleported experimentally in a purely photonic experiment [4]. Single-excitation entanglement in atomic ensembles has also been used to implement the basic segment of a quantum repeater [5]. Note that in principle single-photon states of the form $|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B$ can furthermore be converted into two-atom entangled states of the form $|e\rangle_A|g\rangle_B + |g\rangle_A|e\rangle_B$, cf. [1], which are also often used in quantum communication schemes [6].

Entanglement purification is an important concept in quantum information. It was introduced in Ref.[7], which showed that it is possible to convert two copies of a less entangled state into one copy of a more entangled state using only local operations and classical communication. The first entanglement purification protocols [7, 8] were formulated in terms of qubits and quantum gates, in particular they required CNOT gates. Both for practical applications and from a conceptual point of view it is of great interest to look for implementations that are as simple as possible. For example, Ref. [9] proposed a method for the purification of polarization-entangled photon pairs that could be realized with linear optical elements (without the need for CNOT gates). This procedure was adapted to parametric down-conversion sources in Ref. [10], leading to an experimental realization [11].

An important domain of application for entanglement purification is in the context of long-distance quantum communication. The direct distribution of quantum states is limited by the problem of photon loss in transmission. This can be overcome using quantum repeater protocols [12] based on the creation and storage of entanglement for moderate-distance elementary links, followed

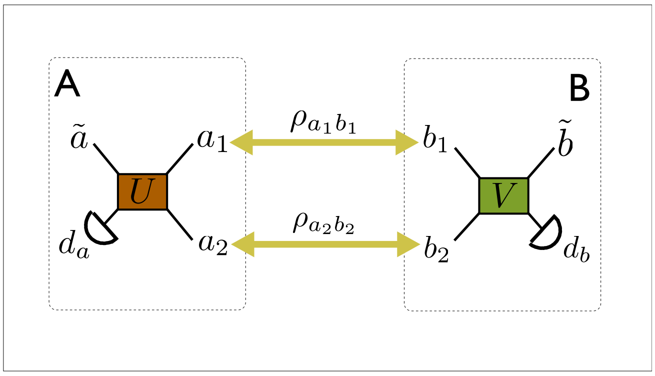

FIG. 1: (Color online) Scheme for entanglement purification of single- photon entanglement. Alice and Bob share two entangled single-photon states $\rho_{a_1b_1}$, $\rho_{a_2b_2}$ with fidelity F. Both parties apply local unitary linear-optical transformations U and V to their respective modes. For appropriately chosen U and V, the detection of one photon in either d_a or d_b projects modes $\tilde{a}$ and $\tilde{b}$ into a single-photon entangled state with higher fidelity.

by entanglement swapping. In practice these operations cannot be performed with perfect fidelity, limiting the number of links that can be used. This number and thus the achievable distance can be greatly increased if entanglement purification is used.

The linear-optics purification protocol of Ref. [9] can be readily integrated into quantum repeater protocols that are based on photon-pair entanglement [13, 14]. However, there are other very attractive protocols based on single-photon detections [15, 16], see e.g. Ref. [5] for a recent related experiment. While being slower than the best known protocol [14], they are rather simple and require significantly fewer resources to outperform the direct transmission of photons. They will thus be easier to implement in the short term, and may well be the first repeater protocols achieving a genuine advantage compared to direct transmission. (More detailed resource counts are given in Ref. [14].) As a consequence of their relative simplicity, protocols based on single-photon detections are also significantly less sensitive to imperfections, such as non-unit memory read out or photon detection efficiencies, as compared with protocols based

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on two-photon detections. The main drawback of protocols based on single-photon detections is that, unlike protocols based on two-photon detections, they are interferometrically sensitive to path length fluctuations [17]. Purification of single-photon entanglement is thus particularly important in this context. However, to our knowledge, no purification procedure for single-photon entanglement has been proposed so far.

Here we show that single-photon entanglement purification can be realized in a very simple way using only linear optical elements and photon detectors. Suppose that Alice and Bob ideally want to share a maximally entangled state $\psi_+^{ab} = \frac{1}{\sqrt{2}}\left(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B\right) = \frac{1}{\sqrt{2}}\left(a^\dagger + b^\dagger\right)|0\rangle$, but that there are phase errors due e.g. to channel length fluctuations, such that without entanglement purification they can only distribute copies of the state

$$\rho_{ab} = F|\psi_+^{ab}\rangle\langle\psi_+^{ab}| + (1-F)|\psi_-^{ab}\rangle\langle\psi_-^{ab}|, \tag{1}$$

with $\frac{1}{2} < F < 1$, where $\psi_-^{ab} = \frac{1}{\sqrt{2}}(a^\dagger - b^\dagger)|0\rangle$ is the state after a phase error [18]; $F = \frac{1}{2}$ corresponds to the case where all phase information has been lost and no entanglement is left. Note that losses and phase errors are the most significant practical limitations in the present context. In the context of quantum repeaters, vacuum ($|0_A\rangle|0_B\rangle$) and multi-photon ($|1_A\rangle|1_B\rangle$) components are also relevant errors. However, vacuum components do not decrease the fidelity of the distributed state since the final measurement in schemes based on single-photon detections post-selects the cases where there was a photon in the output. For multi-photon components, one can use a specific architecture based on single-photon sources which does not create multi-photon components in the elementary link, making it very efficient [16].

We now show how, starting from two copies of the state Eq. (1), one can create one copy of a state of the same form with fidelity $\tilde{F} > F$. We look for a purification protocol that has the simple form shown in Fig. 1. Alice and Bob both perform a linear unitary transformation on their two modes (a_1 and a_2, and b_1 and b_2 respectively), and then they each detect one of the output modes (d_a and d_b). The goal is to have a higher-fidelity single-photon entangled state in modes $\tilde{a}$ and $\tilde{b}$, cf. Fig. 1. The events of interest are thus those where one photon is detected in either d_a or d_b, leaving the other photon in the modes $\tilde{a}, \tilde{b}$. The most general form for the matrix describing Alice's transformation is

$$U(\theta,\phi,\xi) = \begin{bmatrix} \cos\theta e^{i\xi} & -\sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta e^{-i\xi}, \end{bmatrix} \tag{2}$$

with $\{a_1^\dagger|0\rangle, a_2^\dagger|0\rangle\} = U\{\tilde{a}^\dagger|0\rangle, d_a^\dagger|0\rangle\}$. Bob can choose a different transformation, denoted $V(\theta',\phi',\xi')$ which is obtained from U by replacing the arguments by (θ',ϕ',ξ').

We now determine the state of the output modes $\tilde{a}$ and $\tilde{b}$ conditional on the detection of a single photon in mode d_a and of zero photons in mode d_b. The initial state $\rho_{a_1b_1}\otimes\rho_{a_2b_2}$ can be seen as a probabilistic mixture of the four pure states $\psi_+^{a_1b_1}\otimes\psi_+^{a_2b_2}$, $\psi_-^{a_1b_1}\otimes\psi_+^{a_2b_2}$, $\psi_+^{a_1b_1}\otimes\psi_-^{a_2b_2}$, and $\psi_-^{a_1b_1}\otimes\psi_-^{a_2b_2}$ with probabilities F^2, $F(1-F)$, $F(1-F)$, and $(1-F)^2$ respectively. These states give rise to conditional states $\frac{1}{2}(A\tilde{a}^\dagger + B_-\tilde{b}^\dagger)|0\rangle$, $\frac{1}{2}(A\tilde{a}^\dagger - B_+\tilde{b}^\dagger)|0\rangle$, $\frac{1}{2}(A\tilde{a}^\dagger + B_+\tilde{b}^\dagger)|0\rangle$, and $\frac{1}{2}(A\tilde{a}^\dagger - B_-\tilde{b}^\dagger)|0\rangle$ respectively, where $A = \cos 2\theta$, and $B_\pm = \cos\theta e^{-i\xi}\cos\theta' e^{i\xi'} \pm \sin\theta e^{-i\phi}\sin\theta' e^{i\phi'}$ are functions of the matrix elements of U and V. Based on these formulas one can calculate the trace of the conditional state $\tilde{\rho}_{ab}$,

$$\mathrm{tr}\tilde{\rho}_{ab} = \frac{A^2}{4} + \frac{F^2+(1-F)^2}{4}|B_-|^2 + \frac{F(1-F)}{2}|B_+|^2, \tag{3}$$

and its fidelity with respect to the desired state ψ_+^{ab},

$$\tilde{F} = \frac{1}{2} + \frac{A(F^2-(1-F)^2)\mathrm{Re}(B_-)}{A^2 + (F^2+(1-F)^2)|B_-|^2 + 2F(1-F)|B_+|^2}. \tag{4}$$

The case of one photon detected in d_b and no photon in d_a leads to analogous expressions, it is sufficient to exchange the roles of the matrix elements of U and V (i.e. primed and unprimed angles).

Our goal is to find the transformations U and V that maximize the output fidelity Eq. (4). One can see that the optimal fidelity is achieved for $B_+ = 0$ and $\mathrm{Im}(B_-) = 0$. Both these conditions can be satisfied choosing $\xi = \xi' = \phi = \phi' = 0$ and $\theta' = \theta - \frac{\pi}{2}$, giving $B_- = \sin 2\theta$. Due to the above mentioned symmetry, this choice also maximizes the output fidelity for a detection in d_b. Note that the optimum choice of transformations U and V is asymmetric, i.e. $U \neq V$. The fidelity now depends on the single parameter θ,

$$\tilde{F} = \frac{1}{2} + \frac{\left(F^2-(1-F)^2\right)\tan 2\theta}{1+\left(F^2+(1-F)^2\right)\tan^2 2\theta}. \tag{5}$$

For situations where the fidelity of the initial states is known, one can use Eq. (5) to optimize the parameter θ as a function of F. One finds

$$\tan 2\theta_{opt} = \frac{1}{\sqrt{F^2+(1-F)^2}}, \tag{6}$$

giving the optimized output fidelity

$$\tilde{F}_{opt} = \frac{1}{2} + \frac{F^2-(1-F)^2}{2\sqrt{F^2+(1-F)^2}}, \tag{7}$$

cf. Fig. 2.

One can show that this is the optimal fidelity that can be achieved even if one admits auxiliary input modes

that are initially empty. In this more general case the fidelity is still given by Eq. (4), but A, B_+ and B_- now depend on the relevant coefficients of a larger unitary matrix. However, optimizing Eq. (4) as if A, B_+ and B_- were independent variables, one finds the same optimum expression Eq. (7). This implies that Eq. (7) cannot be improved for any number of auxiliary empty modes. The situation may be different for auxiliary photons.

The success probability for the optimized protocol as described above is $p_{opt} = 2\times\mathrm{tr}\tilde{\rho}_{ab}$ since both cases (detection in d_a, no detection in d_b and vice versa) contribute. One finds

$$p_{opt} = \frac{F^2 + (1-F)^2}{1 + F^2 + (1-F)^2}. \tag{8}$$

For situations where the input fidelity F is not known a priori, one has to choose a value of θ that is independent of F. For example, if F is unknown, but expected to be close to 1, one could choose the value of θ_{opt} for $F = 1$, which gives $\theta = \frac{\pi}{8}$. For this simplified protocol one finds an output fidelity

$$\tilde{F}_1 = \frac{F^2 + F}{1 + F^2 + (1-F)^2} \tag{9}$$

and a success probability

$$p_1 = \frac{1 + F^2 + (1-F)^2}{4}, \tag{10}$$

which is close to 1/2 for F close to 1. Fig. 2 shows the output fidelities $\tilde{F}_{opt}$ and $\tilde{F}_1$ both for the optimal protocol and for the simplified protocol. Both protocols achieve substantial purification, and the simplified protocol is almost as effective as the optimal one. It is interesting to consider the regime of high fidelities. For $F = 1 - \epsilon$ both $\tilde{F}_{opt}$ and $\tilde{F}_1$ are equal to $1 - \frac{\epsilon}{2} + O(\epsilon^2)$, i.e. the purification protocol divides the error by a factor of 2. Note that in quantum repeater protocols the error is approximately doubled with every level of entanglement swapping. This means that the present protocol has the potential of significantly increasing the number of possible levels, and thus the achievable total distance.

The proposed protocols are very feasible with current technology. For example, for the simplified protocol, Alice's linear operation U simply corresponds to a beam splitter with (amplitude) transmission coefficient $\cos\frac{\pi}{8}$, corresponding to an intensity transmission of 85%, and Bob's operation V to a beam splitter with amplitude transmission coefficient $\cos\frac{-3\pi}{8}$, corresponding to an intensity transmission of 15%. The optimized protocol requires beam splitters with varying transmission depending on the input fidelity F. Note that if the modes a_1 and a_2 (and analogously b_1 and b_2) are converted to the polarization states of a single spatial mode [5], then the required generalized beam splitters are very easy to realize combining wave plates and polarizing beam splitters.

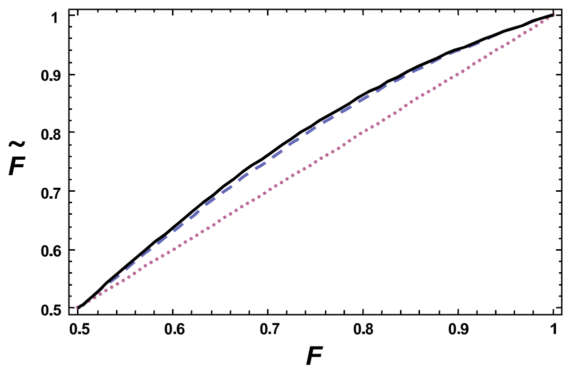

FIG. 2: (Color online) Output fidelity as a function of input fidelity for the optimized protocol, where the parameter θ is adapted to the input fidelity ($\tilde{F}_{opt}$, full), and for a simplified protocol where $\theta = \frac{\pi}{8}$ for all input fidelities ($\tilde{F}_1$, dashed). As a reference the straight line $\tilde{F} = F$ is also shown (dotted).

The protocol relies on single-photon interference and on the bosonic character of indistinguishable photons. Single-photon interference is equivalent to classical interference and can be performed with extremely high precision. The most significant experimental challenge is likely to be the generation of highly indistinguishable photons. Their degree of indistinguishability can be quantified by their mode overlap, which corresponds experimentally to the visibility V of the "Hong-Ou-Mandel dip" [19], i.e. the extent to which the two photons "bunch" after a 50/50 beam splitter ($V = 1$ corresponds to perfect bunching). Ref. [20] has reported a very impressive visibility $V = 0.994$, which is largely sufficient for high-fidelity purification, cf. Fig. 3. The result of Ref. [20] was achieved for photons from the same pair for a parametric down-conversion source, which is likely to be the most promising system for initial demonstrations of the proposed protocol. In general, Fig. 3 shows that purification is possible for dip visibilities that should be achievable in a wide range of systems.

The protocol in its ideal form requires highly efficient photon-number resolving single-photon detectors. We now discuss the impact of non-perfect detection efficiency. The non-detection of a second photon that is actually present in mode d_a or d_b will lead to a vacuum component in modes $\tilde{a}, \tilde{b}$. However, it will not reduce the fidelity for the single-photon component of the output. If the purification protocol is applied in the context of the quantum repeater schemes of Refs. [15], the vacuum component leads to a lower success probability for the subsequent entanglement swapping steps and thus to a lower entanglement generation rate. On the other hand, the fidelity of the generated long-distance entanglement

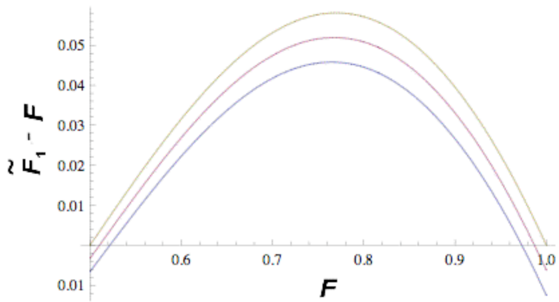

FIG. 3: Fidelity improvement $\tilde{F}_1 - F$ as a function of F for Hong-Ou-Mandel dip visibilities $V = 1, 0.99$ and 0.98 (top to bottom).

only depends on the fidelity of the single-photon component (and thus is independent of the detection efficiency), since the final measurement in these schemes post-selects the cases where there was a photon in the output. Note that highly efficient detectors are being developed, e.g. Ref. [21] has recently reported photon-number resolving detectors with 95% efficiency at 1556 nm, a wavelength ideally suited for long-distance transmission in optical fibers.

The proposed scheme could be demonstrated both in purely photonic experiments, and in experiments involving quantum memories for photons, as an important further step towards the realization of a quantum repeater. The first type of experiment, in addition to the components discussed above, just requires two sources of single photons, which could be realized conditionally based on parametric down-conversion [22] or directly using quantum dots [23] or single atoms [24].

The second type of experiment would be very similar to the experiment of Ref. [5]. Single-particle entanglement would first be created between two pairs of atomic ensembles. The stored excitations are then reconverted into photons, combined on linear optical elements as described above and detected. The conditions for the successful realization of the purification scheme proposed here, in particular indistinguishability of the photons emitted by different ensembles, are similar as for the experiment of Ref. [5]. Interesting further experimental steps would be to re-absorb the purified delocalized photon in a memory, and of course to increase the distance between the two parties.

In conclusion, we have presented a very simple scheme for the purification of single-photon entanglement which is realizable with current technology. We have furthermore shown that the scheme achieves the optimal fidelity for any number of auxiliary vacuum modes. We find the simplicity of the scheme remarkable from a conceptual point of view. It also constitutes important progress for the implementation of quantum repeaters based on single-particle entanglement [15].

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